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www.elsevier.com/locate/jmaaA \mathbb{B} -convex production model for evaluating performance of firmsWalter Briec^{a,*}, Charles Horvath^b^a University of Perpignan, Department of Economics, 52 avenue Paul Alduy, 66800 Perpignan, France^b University of Perpignan, Department of Mathematics, 52 avenue Paul Alduy, 66800 Perpignan, France

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ABSTRACT

Some 30 years ago Charnes, Cooper and Rhodes [A. Charnes, W.W. Cooper, E. Rhodes, Measuring the efficiency of decision making units, *European J. Oper. Res.* 2 (6) (1978) 429–444] proposed *DEA* (Data Envelopment Analysis) as a mean of measuring and evaluating performance of firms. This paper proposes a model for production technologies which differs from the traditional *DEA* production model. The usual convex framework of the *DEA* model is replaced by an order theoretical condition: if two input vectors can produce a given output then the maximum coordinatewise of these two vectors can produce that same output. In this model, technologies are dually linked by a min–max cost function that is dual to the Shephard's distance function. Assuming free disposal of outputs these technologies can be completely described and the Shephard's distance function can be given in closed form.

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1. Introduction

In microeconomic production theory, a technology is characterized by the set of all technically feasible combinations of output and inputs.

Building on the seminal ideas of Farrell [8], Charnes, Cooper and Rhodes [5] proposed to model production technologies using a nonparametric approach that does not involve a functional form of the production set. Among other things, they showed how to determine the efficient observed production units in a sample of firms operating on a specific sector of the economy. They termed their model *DEA* (Data Envelopment Analysis). Banker, Charnes and Cooper [2] extended this approach to the case of variable returns to scale.

While convexity has been traditionally invoked in Operations Research literature on *DEA*, its use is sometimes questionable. Prices are often lacking in the public and privates sectors, performance gauging is necessarily limited to technical rather than allocative efficiency.

One of the assumptions of the model developed in this paper is that the least upper bound of a pair of input vectors can produce the upper bound of the outputs they can individually produce. This model allows a dual economic interpretation of technologies through a so-called min–max cost function. We prove that the traditional Shephard's distance function is primal to this min–max cost function that is dual to the Shephard's distance function. By replacing the usual convex hull of a subset of \mathbb{R}_+^d by what we call its \mathbb{B} -convex hull (\mathbb{B} -convexity has been introduced by Briec and Horvath in [3]), the Shephard distance function can be explicitly calculated.

The paper unfolds as follows. Section 2.1 is for notations. Gauges and cogauges associated to a multivalued map and the duality relation between the cogauge and the “max”-support functional of an upward set are presented in Sections 2.2 and 2.3. The classical *DEA* method for production models is described in Section 3. \mathbb{B} -convexity is presented in Section 4.1; technologies whose input and output sets are semilattices are described in Sections 4.2 and 4.3 as well as the general form

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of their Shephard's distance function; the \mathbb{B} -convex estimation of a technology is introduced in Section 4.4; the Shephard's distance functions for those technologies are explicitly computed in Section 4.5.

2. Gauge, duality and upward sets

2.1. Notations

The set $\mathbb{R}_+^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d : \min\{x_1, \dots, x_d\} \geq 0\}$ is the positive cone of \mathbb{R}^d and \mathbb{R}_{++}^d the set of those elements $(x_1, \dots, x_d) \in \mathbb{R}^d$ such that $\min\{x_1, \dots, x_d\} > 0$. The partial order on \mathbb{R}^d associated to the positive cone \mathbb{R}_+^d is defined, as usual, by $z \leq w$ if $w - z \in \mathbb{R}_+^d$ or, equivalently, $z \leq w$ if, for all $i \in \{1, \dots, d\}$, $z_i \leq w_i$. If $x \leq y$ we denote by $[x, y]$ the set $\{w \in \mathbb{R}^d : x \leq w \leq y\}$. Given u and v in \mathbb{R}^d the vector $u \vee v$ is the least upper bound of u and v , that is, $(\max\{u_1, v_1\}, \dots, \max\{u_d, v_d\})$; $\bigvee_{i=1}^k x^i$ stands for the least upper bound of set of vectors $\{x^1, \dots, x^k\}$ (to avoid confusion between a set of vectors and the coordinates $\{x_1, \dots, x_d\}$ of a given vector x we will denote “the vector of index i ” by x^i).

The norm of a vector $x \in \mathbb{R}^d$ is $\|x\| = \max\{|x_1|, \dots, |x_d|\}$. The *carrier* of an element u of \mathbb{R}_+^m is the set $\text{car}(u) = \{i : u_i > 0\}$.

A multivalued map T from a set X to a set Y assigns to each element x of X a subset $T(x)$ of Y ; no special notation will be used for multivalued or single valued maps, both will simply be called maps (from X to Y). As usual a map $T : X \rightarrow Y$ will be identified with its graph, that is $\{(x, y) \in X \times Y : y \in T(x)\}$. To an arbitrary map $T : X \rightarrow Y$ one associates its inverse T^{-1} ; it is the map from Y to X defined by $x \in T^{-1}(y)$ if and only if $y \in T(x)$. The sets $T(x)$ are the *values* of T while the sets $T^{-1}(y)$ are the *fibers* of T . The image of a subset A of X by T is the set $T(A) = \bigcup_{x \in A} T(x)$.

2.2. Gauges and cogauges

For $x \in \mathbb{R}_+^d$ let $\mathbb{R}_+(x) = \{\lambda x : \lambda \geq 0\}$ and $\mathbb{R}_{++}(x) = \{\lambda x : \lambda > 0\}$; if $x \neq 0$ they are, respectively, the halfline and the open halfline through the point x . The *Minkowski gauge* μ_A and *cogauge* ν_A of a nonempty subset A of \mathbb{R}^d are the maps from \mathbb{R}^d to $\mathbb{R}_+ \cup \{+\infty\}$ respectively defined by

$$\mu_A(x) = \inf\{\lambda > 0 : x \in \lambda A\} \quad \text{and} \quad \nu_A(x) = \sup\{\lambda > 0 : x \in \lambda A\}. \quad (2.1)$$

If $A \cap \mathbb{R}_{++}(x) = \emptyset$ then $\mu_A(x) = \infty$ and $\nu_A(x) = 0$; with this remark one can extend the definition of μ_A and ν_A to the case $A = \emptyset$.

A subset A of \mathbb{R}^d is *radiant* if, for all $0 < \lambda \leq 1$ and all $x \in A$, $\lambda x \in A$; it is *coradiant* if, for all $\lambda \geq 1$ and all $x \in A$, $\lambda x \in A$. A set A is simultaneously radiant and coradiant if and only if, for all $x \in A$, $\mathbb{R}_{++}(x) \subset A$. Such a set is a cone. A closed radiant set is *starshaped*,¹ that is, if $\lambda \in [0, 1]$ and $x \in A$ then $\lambda x \in A$.

One can very easily see that the gauge μ_A , respectively the cogauge ν_A , of an arbitrary radiant, respectively coradiant, set A of \mathbb{R}_+^d is positively homogeneous of degree one.

Theorem 2.2.1.

(1) If A is a closed radiant set of \mathbb{R}_+^d then

$$A = \{x \in \mathbb{R}_+^d : \mu_A(x) \leq 1\}.$$

(2) If A is a closed coradiant set of \mathbb{R}_+^d then $A = \{x \in \mathbb{R}_+^d : \nu_A(x) \geq 1\}$.

This follows from the much more general Propositions 5.1 and 5.6 of [11].

Maps which are cogauges of closed coradiant sets are characterized by the following proposition which can be obtained from Proposition 5.8 of [11].

Theorem 2.2.2. Given a map $q : \mathbb{R}_+^d \rightarrow [0, \infty]$ let $B = \{x \in \mathbb{R}_+^d : q(x) \geq 1\}$. Then statements (1) and (2) are equivalent:

- (1) q is upper semicontinuous, positively homogeneous of degree one and $q \neq 0$;
- (2) B is a nonempty closed coradiant subset of \mathbb{R}_+^d and $\nu_B = q$.

Theorem 2.2.1 tells us how T can be reconstructed from either the gauge or the cogauge:

¹ Starshaped at 0 would be more precise; we will consider only that particular cases.

Proposition 2.2.3.

(1) If T has closed and radiant values then, for all $x \in \mathbb{R}_+^m$,

$$T(x) = \{y \in \mathbb{R}_+^n : \mu_{T(x)}(y) \leq 1\}. \quad (2.2)$$

(2) If T has closed and coradiant fibers then, for all $y \in \mathbb{R}_+^n$,

$$T^{-1}(y) = \{x \in \mathbb{R}_+^m : \nu_{T^{-1}(y)}(x) \geq 1\}. \quad (2.3)$$

2.3. Duality and upward sets

A subset A of \mathbb{R}_+^m is *upward* if the following condition holds:

$$\forall x \in A \quad \forall x' \in \mathbb{R}_+^m \quad [x \leq x' \Rightarrow x' \in A]. \quad (2.4)$$

One will find in Section 3 a production model for technologies with strongly disposable inputs or, said differently, a production model for technologies whose input sets are upward. It is therefore appropriate to begin this section with a short analysis of upward subsets of \mathbb{R}_+^m .

A symmetrical definition is useful to model strongly disposable outputs. A subset B of \mathbb{R}_+^m is *downward* if the following condition holds:

$$\forall y \in B \quad \forall y' \in \mathbb{R}_+^m \quad [y \geq y' \Rightarrow y' \in B]. \quad (2.5)$$

A detailed study of *downward* sets can be found in [10].

Lemma 2.3.1 below gives an analytic characterization of closed upward subsets of \mathbb{R}_+^m . First, let us introduce the following notation²: for $(w, x) \in \mathbb{R}_+^m$ let

$$\langle w, x \rangle = \max_{1 \leq i \leq m} w_i x_i \quad (2.6)$$

and for $A \subset \mathbb{R}_+^m$ let $\sigma_A^{\max} : \mathbb{R}_+^m \rightarrow [0, \infty]$ be the map defined as follows:

$$\sigma_A^{\max}(w) = \inf_{x \in A} \langle w, x \rangle \quad (2.7)$$

if $A \neq \emptyset$ and $\sigma_A^{\max}(w) = +\infty$ otherwise.

For all $u \in \mathbb{R}_+^m$ let $\mathbb{K}(u) = \{v \in \mathbb{R}_+^m : \langle u, v \rangle > 0\}$ and $\mathbb{K}_1(u) = \{v \in \mathbb{R}_+^m : \langle u, v \rangle \geq 1\}$.

Lemma 2.3.1. For all nonempty closed subsets A of $\mathbb{R}_+^m \setminus \{0\}$ the following statements are equivalent:

- (1) A is an upward subset of \mathbb{R}_+^m ;
- (2) for all $x \in \mathbb{R}_+^m \setminus A$ there exists $\hat{w} \in \mathbb{R}_+^m$ such that $\langle \hat{w}, x \rangle < \sigma_A^{\max}(\hat{w})$;
- (3) $A = \{x \in \mathbb{R}_+^m : \forall w \in \mathbb{E}(x), \langle w, x \rangle \geq \sigma_A^{\max}(w)\}$ where $\mathbb{E}(x)$ can be any one of the following sets: $\mathbb{R}_{++}^m, \mathbb{R}_+^m, \mathbb{K}(x), \mathbb{K}_1(x)$.

Proof. (1) Assume that A is closed and upward and that $x \notin A$. Then $\inf_{a \in A} \max_{1 \leq i \leq n} |x_i - a_i| > 0$ and therefore one can choose $\eta > 0$ such that $\hat{x} = (x_1 + \eta, \dots, x_m + \eta) \notin A$. Let $\hat{w} = \hat{x}^{-1}$. We obviously have $\langle \hat{w}, x \rangle \leq 1$. From $\hat{x} \notin A$ and A being upward we have $[0, \hat{x}] \cap A = \emptyset$, that is, for all $a \in A$ there exists at least one index j such that $a_j > \hat{x}_j$ or, equivalently, $\langle \hat{w}, a \rangle > 1$ and therefore $\sigma_A^{\max}(\hat{w}) \geq 1$. This proves (2). Assume that (2) holds then

$$\mathbb{R}_+^m \setminus A \subset \bigcup_{w \in \mathbb{R}_+^m} \{x \in \mathbb{R}_+^m : \langle w, x \rangle < \sigma_A^{\max}(w)\}.$$

The reverse inclusion trivially holds. This proves (3) for $\mathbb{E}(x) = \mathbb{R}_{++}^m$ and $\mathbb{E}(x) = \mathbb{R}_+^m$, since $\mathbb{R}_{++}^m \subset \mathbb{R}_+^m$.

The set defined in (3) is clearly closed and upward. We have shown that (1)–(3) for $\mathbb{E}(x) = \mathbb{R}_{++}^m$ and $\mathbb{E}(x) = \mathbb{R}_+^m$ are equivalent.

To obtain the remaining equivalences notice that if $\langle w, x \rangle > 0$ then $\hat{w} = \langle w, x \rangle^{-1} w \in \mathbb{K}_1(x)$ and use the homogeneity of σ_A and of $w \mapsto \langle w, x \rangle$. \square

² $\langle w, x \rangle_{\max}$, or even $\langle w, x \rangle_{\max, m}$ would be more precise, but the lack of subscript should not be a cause of confusion since the meaning of $\langle w, x \rangle$ will remain constant throughout this paper.

Corollary 2.3.2. *If A is a closed nonempty upward subset of $\mathbb{R}_+^m \setminus \{0\}$ then*

$$A = \left\{ x \in \mathbb{R}_+^m : \inf_{w \in \mathbb{K}_1(x)} \frac{\langle w, x \rangle}{\sigma_A^{\max}(w)} \geq 1 \right\} \quad (2.8)$$

and also

$$A = \left\{ x \in \mathbb{R}_+^m : \sup_{\substack{w \geq 0 \\ \langle w, x \rangle = 1}} \sigma_A^{\max}(w) \leq 1 \right\}. \quad (2.9)$$

Proof. Since $0 \notin A$, we have $\sigma_A^{\max}(w) > 0$ for all $w \neq 0$. Eq. (2.9) is a direct consequence of (2.8) (which does not need a proof), of the homogeneity of the functions involved and of $\langle w_x, x \rangle = 1$ if $\langle w, x \rangle > 0$ and $w_x = \langle w, x \rangle^{-1} w$. \square

A set A_0 is a generating family of the upward set $A \subset \mathbb{R}_+^m$ if, for all $x \in \mathbb{R}_+^m$, $x \in A$ if and only if there exists $a \in A_0$ such that $a \leq x$.

Proposition 2.3.3. *If A_0 is a generating family of the nonempty upward set $A \subset \mathbb{R}_+^m \setminus \{0\}$ then*

$$\nu_A(x) = \sup_{a \in A_0} \min_{i \in \text{car}(a)} \left(\frac{x_i}{a_i} \right) \quad (2.10)$$

and

$$\sigma_A^{\max}(w) = \inf_{x \in A_0} \max_{i \in \text{car}(w)} w_i x_i. \quad (2.11)$$

Proof. Since $0 \notin A$, for all $a \in A_0$ $\text{car}(a) \neq \emptyset$. Let $q(x)$ be the expression on the right-hand side of (2.10). It is a continuous positively homogeneous map that is not identically 0. The inequality $x \geq a$ is equivalent to, for all $i \in \text{car}(a)$, $x_i \geq a_i$. From Theorem 2.2.1, q is ν_A . Eq. (2.11) trivially follows from the definition of σ_A^{\max} and of $\langle w, x \rangle \geq \langle w, a \rangle$ if $x \geq a$. \square

Corollary 2.3.4 (Duality). *If A is a closed and upward subset of \mathbb{R}_+^m then, for all $x, w \in \mathbb{R}_+^m \setminus \{0\}$,*

$$\nu_A(x) = \inf_{w \in \mathbb{K}_1(x)} \frac{\langle w, x \rangle}{\sigma_A^{\max}(w)} = \left[\sup_{\substack{w \geq 0 \\ \langle w, x \rangle = 1}} \sigma_A^{\max}(w) \right]^{-1} \quad (2.12)$$

and

$$\sigma_A^{\max}(w) = \inf_{x \in \mathbb{K}_1(w)} \frac{\langle w, x \rangle}{\nu_A(x)} = \left[\sup_{\substack{w \geq 0 \\ \langle w, x \rangle = 1}} \nu_A(x) \right]^{-1}. \quad (2.13)$$

Proof. Suppose that $A = \emptyset$. Since for all $x, w \in \mathbb{R}_+^m \setminus \{0\}$, $\nu_A(x) = 0$ and $\sigma_A^{\max}(w) = +\infty$, Eqs. (2.12) and (2.13) hold true. Suppose that A is a nonempty set. If $0 \in A$ then $A = \mathbb{R}_+^m$. By definition $\nu_A(x) = +\infty$ and since σ_A^{\max} is identically 0, the result is immediate. If $0 \notin A$ then the first part of equation follows from Corollary 2.3.2 and Theorem 2.2.1; the second part from the homogeneity of the maps involved.

To establish (2.13) notice that by homogeneity Eq. (2.12) can also be written as

$$\nu_A(x) = \inf_{\substack{w \geq 0 \\ \langle w, x \rangle > 0}} \frac{\langle w, x \rangle}{\sigma_A^{\max}(w)}$$

from which we have

$$\sigma_A^{\max}(w) \leq \inf_{\substack{w \geq 0 \\ \langle w, x \rangle > 0}} \frac{\langle w, x \rangle}{\nu_A(x)},$$

let $(a^n)_{n \in \mathbb{N}}$ be a sequence of elements of A such that $\lim_{n \rightarrow \infty} \langle w, a^n \rangle = \sigma_A^{\max}(w)$. From $\nu_A(a^n) \geq 1$ and $\langle w, a^n \rangle > 0$ we obtain $\inf_{\substack{w \geq 0 \\ \langle w, x \rangle > 0}} \frac{\langle w, x \rangle}{\nu_A(x)} \leq \langle w, a^n \rangle$. \square

Corollary 2.3.4 could also has been obtained directly from Proposition 2.3.3. Notice that if A and B are two subsets of \mathbb{R}_+^m satisfying the conditions of Corollary 2.3.4, then $\sigma_A^{\max} = \sigma_B^{\max}$ if and only if $A = B$. One can go a bit further to establish a link between the support function of A and the cogauge of its dual defined by $A^* = \bigcap_{x \in A} \mathbb{K}_1(x)$.

Corollary 2.3.5. *If A is a closed nonempty and upward subset of \mathbb{R}_+^m such that $0 \notin A$ then*

$$\nu_{A^*} = \sigma_A^{\max} \quad \text{and} \quad \nu_A = \sigma_{A^*}^{\max}. \quad (2.14)$$

Proof. By definition, $\mathbb{K}_1(x) = \{w \in \mathbb{R}_+^m : \langle w, x \rangle \geq 1\}$. From $\nu_{A^*} = \mu_{\mathbb{R}_+^m \setminus A^*}$ one has

$$\nu_{A^*}(w) = \inf \left\{ \lambda > 0 : \frac{w}{\lambda} \in \bigcup_{x \in A} (\mathbb{R}_+^m \setminus \mathbb{K}_1(x)) \right\}$$

and therefore

$$\nu_{A^*}(w) = \inf_{x \in A} \inf \left\{ \lambda > 0 : \frac{w}{\lambda} \in \mathbb{R}_+^m \setminus \mathbb{K}_1(x) \right\},$$

which can be rewritten as

$$\nu_{A^*}(w) = \inf_{x \in A} \inf \left\{ \lambda > 0 : \left\langle \frac{w}{\lambda}, x \right\rangle < 1 \right\} = \inf_{x \in A} \langle w, x \rangle = \sigma_A^{\max}(w).$$

The proof of the second statement is similar. \square

There are cases when ν_A can be simply and explicitly computed, as Lemma 2.3.6 below shows.

Lemma 2.3.6. *Let $\varphi_j : \mathbb{R}_+^m \rightarrow \mathbb{R}_+$, $j \in \{1, \dots, n\}$, be upper semicontinuous and positively homogeneous maps and let $A = \bigcap_{j=1}^n \{x \in \mathbb{R}_+^m : r_j \leq \varphi_j(x)\}$. Assume that $A \neq \emptyset$ and $0 \notin A$. Then, for all $x \in \mathbb{R}_+^m \setminus \{0\}$*

$$\nu_A(x) = \min_{r_j \neq 0} \frac{\varphi_j(x)}{r_j}. \quad (2.15)$$

Proof. Since $0 \notin A$, $\{j : r_j \neq 0\} \neq \emptyset$. Obviously

$$A = \bigcap_{r_j \neq 0} \left\{ x \in \mathbb{R}_+^m : \frac{\varphi_j(x)}{r_j} \geq 1 \right\} = \{x \in \mathbb{R}_+^m : q(x) \geq 1\},$$

where $q(x)$ stands for the right-hand side of (2.15). Since A is not empty q is not identically zero, it is also upper semicontinuous and positively homogeneous. By Theorem 2.2.1, $q = \nu_A$. \square

3. Production models

In this section we present the basic concepts of production theory in a nonparametric context.

3.1. The basic framework

A *production technology* is the graph $T \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ of a map from \mathbb{R}_+^m to \mathbb{R}_+^n ; the elements $x = (x_1, \dots, x_m)$ of \mathbb{R}_+^m are the *inputs of the technology* and the elements $y = (y_1, \dots, y_n)$ of \mathbb{R}_+^n are its *outputs*. The map $x \mapsto T(x)$ is the *output map*; its inverse is the *input map*. The set $T \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ is the set of all feasible input–output vectors:

$$T = \{(x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^n : x \text{ can produce } y\}.$$

To a given a map $T : \mathbb{R}_+^m \rightarrow \mathbb{R}_+^n$ one associates two single valued maps D_i and D_o from $\mathbb{R}_+^m \times \mathbb{R}_+^n \rightarrow [0, \infty]$, the so-called output and input Shephard's distance functions, defined as follows:

$$D_o(x, y) = \mu_{T(x)}(y) \quad \text{and} \quad D_i(x, y) = \nu_{T^{-1}(y)}(x). \quad (3.1)$$

Under suitable conditions $D_i(x, y)$ can be interpreted as a *measure of efficiency* of the input vector x given the output vector y . For example, under the conditions of part (2) of Proposition 2.2.3, if $D_i(x, y) < 1$ then x cannot produce y and if $D_i(x, y) > 1$ then there exists a $\lambda < 1$ such that $(\lambda x, y) \in T$; all the nonzero coordinates of λx are strictly smaller than those of x and the output y can be produced from the input λx . If $D_i(x, y) = 1$ then, given the output y , the input vector x is efficient.

There are some standard assumptions that are usually made on the production technology, that T is not empty and closed, for example (see Shephard [12] for a taxonomy); some are purely mathematical others have a natural interpretation.

NFL (there is no free lunch)

$$\forall y \in \mathbb{R}_+^n \quad (0, y) \in T \text{ implies } y = 0.$$

NFL means that a positive output cannot be obtained from a null input.

IS (inputs are strongly disposable)

$$\forall y \in \mathbb{R}_+^n \quad T^{-1}(y) \text{ is an upward set.}$$

IS can also be interpreted as a condition on the output map since it is clearly equivalent to the monotonicity of T , that is, if $x \leq x'$ then $T(x) \subset T(x')$; (*IS*) implies that, for all $y \in \mathbb{R}_+^n$, $T^{-1}(y)$ is coradial. It can also be written as

$$\forall x \in \mathbb{R}_+^m \quad T([0, x]) = T(x). \quad (3.2)$$

The interpretation is obvious: if a given output y can be produced from the input x then it can also be produced from a larger input.

OS (outputs are strongly disposable)

$$\forall x \in \mathbb{R}_+^m \quad T(x) \text{ is a downward set.}$$

OS can be seen as a condition on the input map since it clearly says that T^{-1} is decreasing, that is, if $y' \leq y$ then $T^{-1}(y) \subset T^{-1}(y')$. It implies that T has radial images.

Condition *OS* says that less output can always be produced with the same input.

TS (the technology is strongly disposable)

$$\forall (x, y) \in T \quad \text{if } x \leq x' \text{ and } 0 \leq y' \leq y \text{ then } (x', y') \in T.$$

TS is obviously the conjunction of *IS* and *OS*.

Using the so-called free disposal cone, that is $K = \mathbb{R}_+^m \times (-\mathbb{R}_+^n)$, *TS* can be written as $T = (T + K) \cap (\mathbb{R}_+^m \times \mathbb{R}_+^n)$.

Other commonly found conditions are:

$$(CR \quad \forall \lambda \geq 0) \quad (ND \quad \forall \lambda \geq 1) \quad (NI \quad \forall \lambda \in [0, 1]) \quad \lambda T \subset T.^3$$

Clearly, these conditions imply, in the language of the previous section, that T is, respectively, a cone, coradial, radial.

3.2. Estimating the technology from a given data

To estimate the efficiency of each of the input–output vectors of a given a finite set $A = \{(x^1, y^1), \dots, (x^l, y^l)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ of observed input–output vectors the data set A is embedded in a technology T for which D_i can possibly be used as a measure of efficiency and preferably actually computed, using for example mathematical programming techniques.

Following the work by Farrell [8], Charnes, Cooper and Rhodes [5] introduced the *DEA* model. Under a constant return to scale assumption this nonparametric technology is defined by

$$T_c^+ = \{(x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^n : \exists (x', y') \in Cc(A) \text{ s.t. } x \geq x' \text{ and } y \leq y'\} \quad (3.3)$$

which can also be written as $T_c^+ = [Cc(A) + K] \cap [\mathbb{R}_+^m \times \mathbb{R}_+^n]$.⁴ It is not hard to see that T_c^+ is the smallest closed and convex technology containing A for which (*TS*) and (*CR*) hold, that is the smallest closed convex cone of $\mathbb{R}_+^m \times \mathbb{R}_+^n$ containing A for which T is increasing and T^{-1} is decreasing.

Following Banker, Charnes, Cooper [2], the production technology can also be defined as the weakly monotonic convex hull of the observations, that is

$$T_v^+ = \{(x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^n : \exists (x', y') \in Co(A) \text{ s.t. } x \geq x' \text{ and } y \leq y'\}. \quad (3.4)$$

Equivalently, $T_v^+ = [Co(A) + K] \cap [\mathbb{R}_+^m \times \mathbb{R}_+^n]$.

This discussion summarizes the so-called *DEA* method (Data Envelopment Analysis) for which $D_i(x, y)$ is usually computed using linear programming techniques (see Figs. 1 and 2).

There are models in which convexity is relaxed. A classical example is given by the *FDH* approach introduced and developed in [6] (*FDH* stands for “Free Disposal Hull”). The technology is the smallest set containing the data and satisfying a free disposal assumption.

Such production models are said to be nonparametric because they do not rely on an *a priori* functional specification of the production frontier.

In Section 4.4 the data set A will be embedded in a technology whose construction is similar to that of the *DEA* model with the difference that the convex hull of A will be replaced by a nonconvex set. As we will see, the Shephard’s distance is effectively computable.

³ *CR* stands for Constant Return to scale, *ND* for Non-Decreasing return to scale and *NI* for Non-Increasing return to scale.

⁴ The convex hull and the conical convex hull of a set A are denoted $Co(A)$ and $Cc(A)$ respectively.

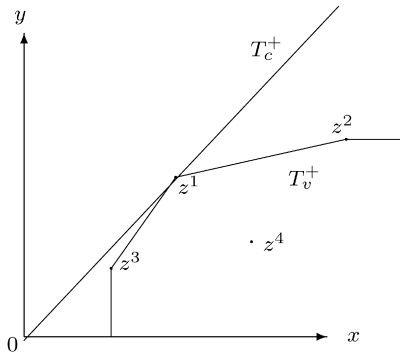


Fig. 1. DEA nonparametric estimation.

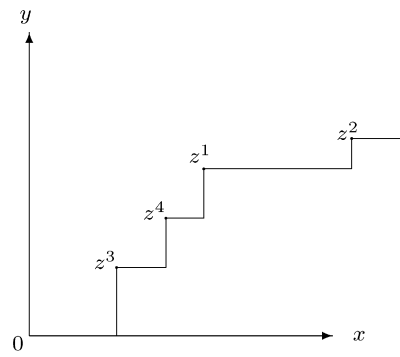


Fig. 2. Free disposal hull of A.

4. \mathbb{B} -convex production technologies

4.1. \mathbb{B} -convexity

In this section we focus on a particular class of path-connected semilattices.

A subset A of \mathbb{R}_+^d is a *semilattice*⁵ if, for all u and v in A , $u \vee v \in A$. The output sets of a technology T are semilattices if, whenever an input x can produce y^1 and y^2 , that is $(x, y^1) \in T$ and $(x, y^2) \in T$, then it can produce $(\max\{y_1^1, y_1^2\}, \dots, \max\{y_n^1, y_n^2\})$. This section describes technologies and their Shephard's distance functions under the assumption that outputs are strongly disposable and input sets are connected semilattices.

This we do by considering the case where the input sets $T^{-1}(y)$ are \mathbb{B} -convex. The \mathbb{B} -convexity assumption means that the technology obeys two basic properties. First it is endowed with an upper semilattice structure: from the least upper bound of two input vectors one can produce the least upper bound of the output vectors they can individually produce. This upper semilattice structure can be seen as fulfilling in an order theoretical framework the role of the additive structure in the traditional convex framework. Moreover \mathbb{B} -convexity implies that the production vectors are divisible as in the usual convexity. We have mentioned above that a \mathbb{B} -convex technology is a path-connected upper semilattice and this makes continuous transformations of production techniques possible which is important in economics.

We say that a subset C of \mathbb{R}_+^d is a \mathbb{B} -convex set if

$$\forall (z^1, z^2, t) \in C \times C \times [0, 1] \quad tz^1 \vee z^2 \in C. \quad (4.1)$$

Remark 4.1.1. A \mathbb{B} -convex set C is a semilattice (take $t = 1$) and it is path-connected.

Remark 4.1.2. If C is a \mathbb{B} -convex subset of $\mathbb{R}_+^m \times \mathbb{R}_+^n$ then its projections on \mathbb{R}_+^m and \mathbb{R}_+^n are also \mathbb{B} -convex.

Remark 4.1.3. If C is an upward subset of \mathbb{R}_+^m then it is \mathbb{B} -convex, since $\forall (z^1, z^2, t) \in \mathbb{R}_+^m \times \mathbb{R}_+^m \times [0, 1]$ we have $tz^1 \vee z^2 \geq z^2$ which implies that $tz^1 \vee z^2 \in C$ if $z^2 \in C$.

For an arbitrary finite subset $A = \{z^1, \dots, z^l\}$ of \mathbb{R}_+^d let

$$\mathbb{B}(A) = \left\{ \bigvee_{k=1}^l t_k z^k : (t_1, \dots, t_l) \in [0, 1]^l \text{ and } \max\{t_1, \dots, t_l\} = 1 \right\}. \quad (4.2)$$

$\mathbb{B}(A)$ is the \mathbb{B} -convex hull of A ; it is clearly compact and path-connected.

Examples of \mathbb{B} -convex sets are depicted in Figs. 3 and 4.

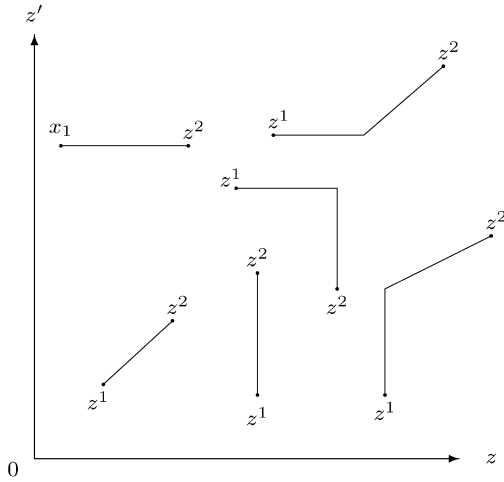
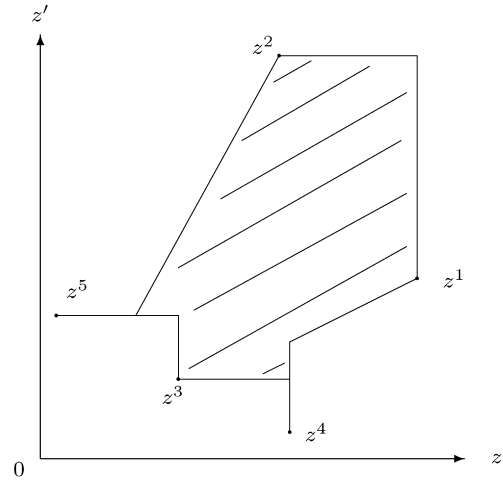
More details on \mathbb{B} -convex sets can be found in Briec and Horvath [3], Briec, Horvath and Rubinov [4] and Adilov and Rubinov [1].

4.2. Technologies whose input sets are semilattices

From Remark 4.1.3 an input set satisfying a free disposal assumption is \mathbb{B} -convex.

Given a technology $T \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$, the *min-max cost function* associated to T is the map C^{\max} from $\mathbb{R}_+^n \times \mathbb{R}_+^m$ to $\mathbb{R}_+ \cup \{+\infty\}$ defined by

⁵ An upper semilattice would be more precise.

Fig. 3. \mathbb{B} -convex hulls of 2 points.Fig. 4. \mathbb{B} -convex hull of 5 points.

$$C^{\max}(w, y) = \sigma_{T^{-1}(y)}^{\max}(w). \quad (4.3)$$

Proposition 4.2.1. Let $T \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ be a technology such that, for all $y \in \mathbb{R}_+^n$, $T^{-1}(y)$ is closed. Then IS holds if and only if, for all $y \in \mathbb{R}_+^n$,

$$T^{-1}(y) = \{x \in \mathbb{R}_+^m : \forall w \in \mathbb{R}_+^m \langle w, x \rangle \geq C^{\max}(w, y)\}. \quad (4.4)$$

Proof. If $0 \in T^{-1}(y)$ then, from IS, $T^{-1}(y) = \mathbb{R}_+^m$. If $0 \notin T^{-1}(y)$ then the conclusion follows from Lemma 2.3.1. \square

The main result of this section, Proposition 4.2.2 below, shows that, under strong disposability of outputs, the Shephard distance function and the min-max cost function are dual to each other.

Proposition 4.2.2. Let $T \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ be a technology such that, for all $y \in \mathbb{R}_+^n$, $T^{-1}(y)$ is closed and upward, that is (IS) holds. Then, for all input vector $x \in \mathbb{R}_+^m \setminus \{0\}$, one has

$$D_i(x, y) = \inf_{w \in \mathbb{K}_1(x)} \frac{\langle w, x \rangle}{C^{\max}(w, y)} \quad (4.5)$$

and

$$C^{\max}(w, y) = \inf_{x \in \mathbb{K}_1(w)} \frac{\langle w, x \rangle}{D_i(x, y)}. \quad (4.6)$$

We show below that the result established in Proposition 4.2.2 can be generalized to technologies with \mathbb{B} -convex input sets.

Intuitively, this means that the producer seeks to find a virtual price minimizing the ratio between his or her cost and the minimum cost that is the cost function. Notice that strong disposability assumption is not required in the next result. However, the ray spanned by the input vector we consider must intersect the input set.

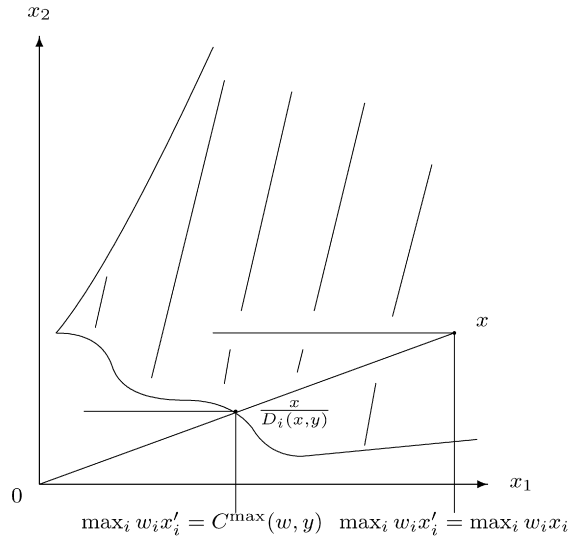
Proposition 4.2.3. Let $T \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ be a technology such that, for all $y \in \mathbb{R}_+^n$, $T^{-1}(y)$ is closed and \mathbb{B} -convex. Then, for all input vector $x \in \mathbb{R}_+^m \setminus \{0\}$, such that $\mathbb{R}_{++}(x) \cap T^{-1}(y) \neq \emptyset$, Eqs. (4.5) and (4.6) hold true.

Proof. Consider the set $T_+^{-1}(y) = T^{-1}(y) + \mathbb{R}_+^m$, which is clearly upward; from Corollary 2.3.4 we have

$$v_{T_+^{-1}(y)}(x) = \inf_{w \in \mathbb{K}_1(x)} \frac{\langle w, x \rangle}{\sigma_{T_+^{-1}(y)}^{\max}(w)} \quad \text{and} \quad \sigma_{T_+^{-1}(y)}^{\max}(w) = \inf_{x \in \mathbb{K}_1(w)} \frac{\langle w, x \rangle}{v_{T_+^{-1}(y)}(x)}.$$

Assuming that $\mathbb{R}_{++}(x) \cap T^{-1}(y) \neq \emptyset$ we show that

$$(i) \quad \sigma_{T^{-1}(y)}^{\max} = \sigma_{T_+^{-1}(y)}^{\max} \quad \text{and} \quad (ii) \quad v_{T^{-1}(y)} = v_{T_+^{-1}(y)}.$$

Fig. 5. Duality and \mathbb{B} -convex input set.

Proof of (i). From $T^{-1}(y) \subset T^{-1}(y) + \mathbb{R}_+^m$ we have $\sigma_{T^{-1}(y)}^{\max} \geq \sigma_{T_+^{-1}(y)}^{\max}$. Moreover, since for all $w \in \mathbb{R}_+^n$, the map $x \mapsto \max_{i=1, \dots, n} w_i x_i$ is nondecreasing we also have

$$\sigma_{T^{-1}(y)}^{\max}(w) = \inf_{x \in T^{-1}(y)} \max_{i=1, \dots, n} w_i x_i \leq \inf_{\substack{x \in T^{-1}(y) \\ u \in \mathbb{R}_+^n}} \max_{i=1, \dots, n} w_i (x_i + u_i) = \sigma_{T_+^{-1}(y)}^{\max}(w).$$

Proof of (ii). If $0 \in T^{-1}(y)$ then $0 \in T_+^{-1}(y)$ and therefore $v_{T_+^{-1}(y)}(x) = v_{T^{-1}(y)}(x) = +\infty$. Suppose that $0 \notin T^{-1}(y)$. In such a case $0 \notin T_+^{-1}(y)$. If d is the distance associated to a given but otherwise arbitrary norm on \mathbb{R}^m we must have $d(0, T_+^{-1}(y)) > 0$, since $T_+^{-1}(y)$ is a closed subset of \mathbb{R}_+^m . For $\lambda \geq \frac{\|x\|}{d(0, T_+^{-1}(y))}$, we have $d(0, \frac{x}{\lambda}) = \|\frac{x}{\lambda}\| \leq d(0, T_+^{-1}(y))$. Consequently, $v_{T_+^{-1}(y)}(x) \leq \frac{\|x\|}{d(0, T_+^{-1}(y))} < +\infty$.

Let $\bar{\rho} = \frac{v_{T^{-1}(y)}(x)}{v_{T_+^{-1}(y)}(x)}$ and set

$$p(x) = [v_{T^{-1}(y)}(x)]^{-1} x \quad \text{and} \quad p_+(x) = [v_{T_+^{-1}(y)}(x)]^{-1} x,$$

respectively. We need to prove that $p_+(x) \in T^{-1}(y)$. Since $T^{-1}(y)$ is closed and $v_{T^{-1}(y)}(x, y) > 0$, we have $p(x) \in T^{-1}(y)$. Moreover, there exists $x^0 \in T^{-1}(y)$ such that $x^0 \leq p_+(x)$. Let

$$\bar{x} = x_0 \vee \bar{\rho} p(x)$$

and notice that $\bar{\rho} p(x) = p_+(x)$ and $\bar{\rho} \leq 1$. Since both x_0 and $p(x)$ belong to $T^{-1}(y)$ and $T^{-1}(y)$ is \mathbb{B} -convex we have $\bar{x} \in T^{-1}(y)$. Finally, from $x_0 \leq p_+(x)$ we have

$$\bar{x} = x_0 \vee p_+(x) = p_+(x) \in T^{-1}(y)$$

and therefore, $v_{T_+^{-1}(y)}(x) \leq v_{T^{-1}(y)}(x)$. \square

The type of input sets depicted in the figure below may be of some importance since they allow for a possible congestion of the technology. This means that there is a lack of the disposability of certain inputs whose use does not necessarily increase the production. This is frequently the case in agricultural and environmental economics (see Fig. 5).

4.3. A class of technologies whose output sets are semilattices

A technology $T \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ is a *Kohli input price (KI) nonjoint technology* if there exist n single output technologies $T^j \subset \mathbb{R}_+^m \times \mathbb{R}_+$ such that, for all input vectors $x \in \mathbb{R}_+^m$, $T(x) = T^1(x) \times \dots \times T^n(x)$. This is a generalization of the fixed-coefficient Leontief transformation. More details can be found in [9].

Let $\{e_1, \dots, e_n\}$ be the canonical basis of \mathbb{R}^n . To a given technology $T \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ we associate n single output technologies defined as follows: $T^{[j]}(x) = \{y_j \in \mathbb{R}_+ : y_j e_j \in T(x)\}$.

Theorem 4.3.1. Let $T \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ be a technology whose output sets are connected semilattices and assume that, for all x , $0 \in T(x)$. Then the following statements are equivalent:

- (1) outputs are freely disposable, that is OS holds;
- (2) for all $x \in \mathbb{R}_+^m$, $T(x) = T^{[1]}(x) \times \cdots \times T^{[n]}(x)$;
- (3) for all $y \in \mathbb{R}_+^n$, $T^{-1}(y) = \bigcap_{j=1}^n T^{[j]-1}(y_j)$.

Proof. If OS holds and if $y \in T(x)$ then, for all $j \in \{1, \dots, n\}$, $y_j e_j$ belongs to $T(x)$; in other words, $y_j \in T^{[j]}(x)$. This shows that

$$T(x) \subset T^{[1]}(x) \times \cdots \times T^{[n]}(x).$$

The reverse inclusion is an obvious consequence of $y = \bigvee_{j=1}^n y_j e_j$ since $T(x)$ is a semilattice. We have shown that (1) implies (2).

Assume that (2) holds. Then, for all $j \in \{1, \dots, n\}$, $T^{[j]}(x)$ is connected, and it contains 0 by hypothesis. In conclusion $T^{[j]}(x)$ is an interval of \mathbb{R}_+ containing 0. If $y \in T(x)$ and if $y' \leq y$ then, for all $j \in \{1, \dots, n\}$, $0 \leq y'_j \leq y_j \in T^{[j]}(x)$ and therefore $y'_j \in T^{[j]}(x)$. We have shown that $y' \in T^{[1]}(x) \times \cdots \times T^{[n]}(x)$ and also that (1) and (2) are equivalent. The equivalence of (2) and (3) is purely set theoretical. \square

We have seen in the course of the proof that $T^{[j]}(x)$ is an interval of \mathbb{R}_+ containing 0. If (2) holds then $T(x)$ is compact if and only if each $T^{[j]}(x)$ is a compact interval of \mathbb{R}_+ containing 0. From this one easily proves the following corollary.

Corollary 4.3.2. Let $T \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ be a technology whose output sets are connected semilattices; assume that, for all x , $0 \in T(x)$ and that OS holds:

- (1) T has compact values if and only if there exist functions $\varphi_j : \mathbb{R}_+^m \rightarrow \mathbb{R}_+$, the production functions of the technology, such that, for all $x \in \mathbb{R}_+^m$, $T(x) = [0, \varphi_1(x)] \times \cdots \times [0, \varphi_n(x)]$, and therefore, for all $y \in \mathbb{R}_+^n$, $T^{-1}(y) = \bigcap_{j=1}^n \{x \in \mathbb{R}_+^m : y_j \leq \varphi_j(x)\}$;
- (2) T has compact values and inputs are freely disposable, IS holds, if and only if all the production functions φ_j are increasing;
- (3) T is a technology with compact values with constant return to scale if and only if, for all $j \in \{1, \dots, n\}$, φ_j superhomogeneous, that is, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+^m$, $\varphi_j(tx) \geq t\varphi_j(x)$;
- (4) If T has compact values and all the value functions are upper semicontinuous then T has closed input sets.

Proposition 4.3.3. Let T be a technology with compact values for which OS, NFL and constant returns to scale hold. For all $j \in \{1, \dots, n\}$, let $D_i^{[j]}$ be the Shephard's distance function of the single output technology $T^{[j]}$. Then, if all the φ_j are upper semicontinuous and positively homogenous maps, one has, for all $x, y \in \mathbb{R}_+^n \setminus \{0\}$ such that $T^{-1}(y) \neq \emptyset$,

$$D_i(x, y) = \min_{j \in \text{car}(y)} \frac{\varphi_j(x)}{y_j} = \min_{j \in \text{car}(y)} D_i^{[j]}(x, y_j). \quad (4.7)$$

Proof. This is a reformulation of Lemma 2.3.6. \square

A few remarks are in order. The results of this section specify the structure of a technology under some structural assumptions on the input and output sets. The production functions themselves are not explicitly given and very little can be said about their nature; as a matter of fact they could be rather arbitrary. Computing the Shephard's distance $D_i(x, y)$ from (4.7) could therefore be a hopeless task. Also one can see from (4.7) that for $D_i(x, y)$ to be equal to 1 it is sufficient to have $D_i^{[j]}(x, y_j) \geq 1$ for all j and $D_i^{[j_0]}(x, y_{j_0}) = 1$ for one index j_0 . In other words, (x, y) is an efficient program if, for all $j \in \{1, \dots, n\}$, (x, y_j) is a feasible program for the technology T^j and, for one j_0 , (x, y_{j_0}) is an efficient program for T^{j_0} .

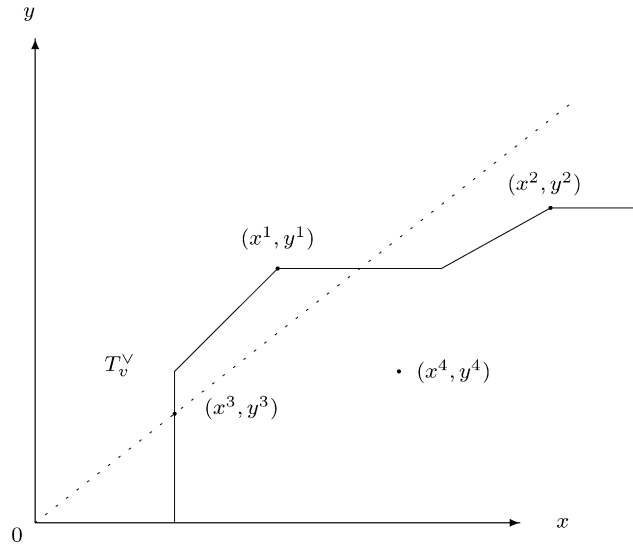
4.4. \mathbb{B} -convex estimation of a technology

To a given data set $A \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ we associate a technology whose values are compact connected semilattices. For these technologies the functions φ_j as well as the Shephard's distance functions can be explicitly computed. The construction is done as with the usual DEA model with the difference that the convex hull of a set A is replaced by what we call its \mathbb{B} -convex hull. Let $A = \{(x^1, y^1), \dots, (x^l, y^l)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ be a set of l observed production vectors. The subset of $\mathbb{R}_+^m \times \mathbb{R}_+^n$ defined by

$$T_v^\vee = \{(x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^n : \exists (x', y') \in \mathbb{B}(A) \text{ s.t. } x \geq x' \text{ and } y \leq y'\} \quad (4.8)$$

is the \mathbb{B} -convex estimation of the production technology under a variable returns to scale assumption. One can equivalently write $T_v^\vee = (\mathbb{B}(A) + K) \cap [\mathbb{R}_+^m \times \mathbb{R}_+^n]$, where $K = \mathbb{R}_+^m \times (-\mathbb{R}_+^n)$.

This type of technology is depicted in Fig. 6.

Fig. 6. \mathbb{B} -convex estimation.

Proposition 4.4.1. For all subsets $A = \{(x^1, y^1), \dots, (x^l, y^l)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ of l observed production vectors, the nonparametric technology T_v^\vee has the following properties:

- (1) T_v^\vee is a closed \mathbb{B} -convex subset of $\mathbb{R}_+^m \times \mathbb{R}_+^n$;
- (2) TS (therefore OS and IS) holds;
- (3) for all $x \in \mathbb{R}_+^m$, $T_v^\vee(x)$ is a compact \mathbb{B} -convex subset of \mathbb{R}_+^n ;
- (4) for all $y \in \mathbb{R}_+^n$, $T_v^{\vee-1}(y)$ is a closed \mathbb{B} -convex subset of \mathbb{R}_+^m .

Proof. (1) From the definition of T_v^\vee and from (4.1) the set T_v^\vee is a \mathbb{B} -convex subset of $\mathbb{R}_+^m \times \mathbb{R}_+^n$. From Remark 4.1.2, the values of T_v^\vee are \mathbb{B} -convex.

The set A is finite, which implies that $\mathbb{B}(A)$ is compact. The set $\mathbb{B}(A) + K$ is the sum of a compact set with a closed set, it is therefore closed. Consequently, $(\mathbb{B}(A) + K) \cap [\mathbb{R}_+^m \times \mathbb{R}_+^n]$ is closed and therefore T_v^\vee is closed.

(2) If $x \leq \hat{x}$ and $y \in T_v^\vee(x)$ then there exists $(x', y') \in \mathbb{B}(A)$ such that $x \geq x'$ and $y \leq y'$; we trivially also have $\hat{x} \geq x'$ which shows that $y \in T_v^\vee(\hat{x})$. In other words $T_v^\vee(x) \subset T_v^\vee(\hat{x})$. Similarly, if $y \leq \hat{y}$ then $T_v^{\vee-1}(\hat{y}) \subset T_v^{\vee-1}(y)$.

(3) $T_v^\vee(x)$ is the projection on \mathbb{R}_+^n of $[\{x\} \times \mathbb{R}_+^n] \cap T_v^\vee$, which is \mathbb{B} -convex in $\mathbb{R}_+^m \times \mathbb{R}_+^n$; by Remark 4.1.2, $T_v^\vee(x)$ is a \mathbb{B} -convex set.

Let us see that $T_v^\vee(x)$ is closed. If $(y^k)_{k \in \mathbb{N}}$ is a sequence of elements of $T_v^\vee(x)$ which converges to y^* then take a sequence $((x^k, y'^k))_{k \in \mathbb{N}}$ in $\mathbb{B}(A)$ such that, for all $k \in \mathbb{N}$, $x \geq x^k$ and $y'^k \geq y^k$. From the compactness of $\mathbb{B}(A)$ we deduce, as in (1), that $y^* \in T_v^\vee(x)$. To complete the proof of this part we verify that $T_v^\vee(x)$ is bounded. If $y \in T_v^\vee(x)$ there exists $y' \in \mathbb{B}(A)$ such that $y \leq y'$. By (4.2) there exists $(t_1, \dots, t_l) \in [0, 1]$ such that $y' = \bigvee_{i=1}^l t_i y_i$ and therefore $y' \leq \bigvee_{i=1}^l y_i$.

The proof of (4) is entirely similar to the proof of (3). \square

Theorem 4.3.1 and Corollary 4.3.2 cannot be directly used for the technology T_v^\vee since the condition $0 \in T_v^\vee(x)$ might not hold. As one can see, $0 \in T_v^\vee(x)$ if and only if $T_v^\vee(x) \neq \emptyset$.

As with the standard DEA model we define the \mathbb{B} -convex estimation of the technology under a constant return to scale assumption by replacing in the definition of T_c^\vee the \mathbb{B} -convex hull of a subset A of \mathbb{R}_+^d , that is $\mathbb{B}(A)$, by its \mathbb{B} -convex conic hull of A , that is

$$\mathbb{B}_c(A) = \left\{ \bigvee_{k=1}^l t_k z^k : (t_1, \dots, t_l) \in \mathbb{R}_+^l \right\}. \quad (4.9)$$

This gives the following technology, which is clearly \mathbb{B} -convex and satisfies a constant return to scale assumption.

$$T_c^\vee = \{(x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^n : \exists (x', y') \in \mathbb{B}_c(A) \text{ s.t. } x \geq x' \text{ and } y \leq y'\}. \quad (4.10)$$

Lemma 4.4.2. If T is a closed production technology then under a constant return to scale assumption the following statements are equivalent:

- (1) *There is no free lunch (NFL);*
- (2) *$T(0)$ is compact;*
- (3) *for all $x \in \mathbb{R}_+^m$, $T(x)$ is compact.*

Proof. Condition (NFL) means that either $T(0) = \emptyset$ or $T(0) = \{0\}$ which clearly implies that $T(0)$ is compact. On the other hand, if (NFL) does not hold there is a nonzero element, call it y^* , in $T(0)$ and therefore, by the constant return to scale assumption, $ty^* \in T(0)$ for all $t > 0$; $T(0)$ is not compact. We have shown that (1) and (2) are equivalent. Obviously, (3) implies (2). To complete the proof let us assume that there exists $x \in \mathbb{R}_+^m$ such that $T(x)$ is not compact and let us show that condition (NFL) does not hold. Since $T(x)$ is not compact and closed (if $y_n \in T(x)$ and if $\lim_{n \rightarrow \infty} y_n = y^*$ then $(x, y_n) \in T$ and $\lim_{n \rightarrow \infty} (x, y_n) = (x, y^*)$ and therefore $(x, y^*) \in T$) it is unbounded. There is a sequence $(y^k)_{k \in \mathbb{N}}$ in $T(x)$ such that $\lim_{k \rightarrow \infty} \|y^k\| = +\infty$, where $\|\cdot\|$ is an arbitrary norm defined on \mathbb{R}^n . We can assume $\|y^k\| > 1$ for all $k \in \mathbb{N}$. Let $x_k = \|y^k\|^{-1}x$ and $y_k = \|y^k\|^{-1}y^k$. Since the sequence $(x_k, y_k)_{k \in \mathbb{N}}$ is bounded it contains a convergent subsequence. Without loss of generality we can assume that the sequence itself converges; let (x^*, y^*) be its limit. Since T satisfies a constant returns to scale assumption we have, for all $k \in \mathbb{N}$, $(x_k, y_k) \in T$, and since T is closed we also have $(x^*, y^*) \in T$. Clearly, $x^* = 0$ and $\|y^*\| = 1$, in other words, $y^* \in T(0)$ and $y^* \neq 0$. \square

Most of the following properties are immediate. The compactness of $T_c^\vee(x)$ for all input vectors x is an immediate consequence of the lemma above.

Corollary 4.4.3. *Let T_c^\vee be the \mathbb{B} -convex estimation of a technology associated to the data $A = \{(x^1, y^1), \dots, (x^l, y^l)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ under a constant return to scale assumption. Then, if T_c^\vee satisfies OS and IS, for all $x \in \mathbb{R}_+^m$, $0 \in T_c^\vee(x)$ and $T_c^\vee(x)$ is \mathbb{B} -convex. Moreover, if for $k = 1 \dots l$ one has $y_k \neq 0$ then T_c^\vee satisfies NFL and $T_c^\vee(x)$ is compact for all $x \in \mathbb{R}_+^m$.*

We close this section with a lemma which reduces the study of T_c^\vee to that of T_v^\vee . Let $1_l \in \mathbb{R}^l$ be the vector whose coordinates are all equal to 1. Let $\{e^1, \dots, e^l\}$ be the elements of the canonical basis of \mathbb{R}^l . Given $A = \{(x^1, y^1), \dots, (x^l, y^l)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ let

$$\tilde{A} = \{((x^1, e^1), (y^1, e^1)), \dots, ((x^l, e^l), (y^l, e^l))\} \subset \mathbb{R}_+^{m+l} \times \mathbb{R}_+^{n+l}. \quad (4.11)$$

Since we will have to consider different data sets we will write \tilde{T}_c^\vee , the production set constructed from the data set \tilde{A} .

Lemma 4.4.4. *Given $A = \{(x^1, y^1), \dots, (x^l, y^l)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$, for all $(x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^n$, $(x, y) \in T_v^\vee$ if and only if there exists $k_0 \in \{1, \dots, l\}$ such that $((x, 1_{l_0}), (y, e^{k_0})) \in \tilde{T}_c^\vee$.*

Proof. By definition of T_v^\vee , $(x, y) \in T_v^\vee$ if and only if there exists $(\rho_1, \dots, \rho_l) \in [0, 1]^l$ such that

$$\max_{1 \leq k \leq l} \rho_k = 1, \quad x \geq \bigvee_{k=1}^l \rho_k x^k \quad \text{and} \quad y \leq \bigvee_{k=1}^l \rho_k y^k \quad (4.12)$$

and, by definition of T_c^\vee , $((x, 1_l), (y, e^{k_0})) \in \tilde{T}_c^\vee$ if and only if there exists $(\rho_1, \dots, \rho_l) \in \mathbb{R}_+^l$ such that

$$(x, 1_l) \geq \bigvee_{k=1}^l \rho_k (x^k, e^k) \quad \text{and} \quad (y, e^{k_0}) \leq \bigvee_{k=1}^l \rho_k (y^k, e^k). \quad (4.13)$$

Notice that (4.13) reduces to $x \geq \bigvee_{k=1}^l \rho_k x^k$, $y \leq \bigvee_{k=1}^l \rho_k y^k$, $0 \leq \rho_k \leq 1$ for all k and $\rho_{k_0} = 1$. \square

4.5. Computing the Shephard's distance

Proposition 4.5.1. *Let T_c^\vee be the \mathbb{B} -convex estimation of a technology associated to the data $A = \{(x^1, y^1), \dots, (x^l, y^l)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ under a constant return to scale assumption and such that there is no free lunch. Let, for all $k \in \{1, \dots, l\}$,*

$$\alpha_k(x) = \min_{i \in \text{car}(x^k)} \frac{x_i}{x_i^k}. \quad (4.14)$$

Then, for all $y \in \mathbb{R}_+^n \setminus \{0\}$ such that $T_c^{\vee-1}(y) \neq \emptyset$,

$$D_i(x, y) = \min_{j \in \text{car}(y)} \frac{\max_{1 \leq k \leq l} \{\alpha_k y_j^k(x)\}}{y_j}. \quad (4.15)$$

Furthermore for all $x \in \mathbb{R}_+^m \setminus \{0\}$,

$$T_c^\vee(x) = [0, \varphi_1(x)] \times \cdots \times [0, \varphi_n(x)], \quad (4.16)$$

where

$$\varphi_j(x) = \max_{1 \leq k \leq l} \{\alpha_k(x) y_j^k\}. \quad (4.17)$$

Proof. Let for all, $k \in \{1, \dots, l\}$,

$$\alpha_k(x) = \min_{i \in \text{car}(x^k)} \frac{x_i}{x_i^k} \quad (4.18)$$

and, for all $j \in \{1, \dots, n\}$,

$$\varphi_j(x) = \max_{1 \leq k \leq l} \{\alpha_k y_j^k(x)\}. \quad (4.19)$$

Since there is no free lunch $\text{car}(x^k) \neq \emptyset$ and $\alpha_k(x)$ is well defined for all $k \in \{1, \dots, l\}$.

Recall that $(x, y) \in T_c^\vee$ if and only if there exists $(\rho_1, \dots, \rho_l) \in \mathbb{R}_+^l$ such that $x \geq \bigvee_{k=1}^l \rho_k x^k$ and $y \leq \bigvee_{k=1}^l \rho_k y^k$. The first inequality says that, for all $i \in \{1, \dots, m\}$ and all $k \in \{1, \dots, l\}$, $x_i \geq \rho_k x_i^k$ which is clearly equivalent to $\alpha_k(x) \geq \rho_k$. If the second inequality holds then $y \leq \bigvee_{k=1}^l \alpha_k(x) y^k$ also holds. The inequality $x \geq \bigvee_{k=1}^l \alpha_k(x) x^k$ always holds. In other words, $(x, y) \in T_c^\vee$ if and only if $y \leq \bigvee_{k=1}^l \alpha_k(x) y^k$ or, equivalently, for all $j \in \{1, \dots, n\}$, $y_j \leq \max_{1 \leq k \leq l} \alpha_k(x) y_j^k$. Lemma 2.3.6 yields the conclusion. \square

Proposition 4.5.2. Let $A = \{(x^1, y^1), \dots, (x^l, y^l)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$. Suppose that there is no free lunch. For all $k \in \{1, \dots, l\}$ and all $(x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^n$ let

$$\alpha_k(x) = \min_{i \in \text{car}(x^k)} \frac{x_i}{x_i^k}, \quad \alpha(x) = \max_{1 \leq k \leq l} \alpha_k(x)$$

and

$$\beta_j(x, y) = \max_{\substack{1 \leq k \leq l \\ y_j \leq y_j^k}} \frac{\alpha_k(x) y_j^k}{y_j}.$$

For all $y \in \mathbb{R}_+^n \setminus \{0\}$ such that $T_v^{\vee-1}(y) \neq \emptyset$ the Shephard's distance function is given by

$$D_i(x, y) = \min \left\{ \min_{j \in \text{car}(y)} \beta_j(x, y), \alpha(x) \right\}. \quad (4.20)$$

Furthermore, for all $x \in \mathbb{R}_+^m$, $T_v^\vee(x) \neq \emptyset$ if and only if $\alpha(x) \geq 1$ and, if $\alpha(x) \geq 1$, then

$$T_v^\vee(x) = \prod_{j=1}^n [0, \varphi_j(x)], \quad (4.21)$$

where

$$\varphi_j(x) = \max_{1 \leq k \leq l} \min \{\alpha_k(x) y_j^k, y_j^k\}. \quad (4.22)$$

Proof. Let \tilde{D}_i be the Shephard's distance function computed on \tilde{T}_v^\vee . Using Lemma 4.4.4 the computation of \tilde{D}_i is reduced to that of \tilde{D}_i . Starting from $(x, y) \in T_v^\vee$ if and only if there exists $k_0 \in \{1, \dots, l\}$ such that $(x + 1_l, y + e^{k_0}) \in \tilde{T}_c^\vee$ we proceed with \tilde{T}_c^\vee as in the proof of Proposition 4.5.1 to find that $(x, y) \in T_v^\vee$ if and only if there exists k_0 such that

$$\min_{j \in \text{car}(y)} \max_{1 \leq k \leq l} \min \left\{ \alpha_k(x) \frac{y_j^k}{y_j}, \frac{y_j^k}{y_j} \right\} \geq 1 \quad (4.23)$$

and

$$\min \{\alpha_{k_0}(x), 1\} \geq 1 \quad (4.24)$$

from which (4.20) easily follows. For the last part, that is (4.21) and (4.22) recall that $T_v^{\vee-1}(y)$ is closed and coradial and use part 2 of Proposition 2.2.3. \square

Table 1
Data sample.

Firms	Input	Output 1	Output 2
1	2	3/2	1
2	2	2	1
3	4	3	2
4	6	6	6
5	7	6	6
6	8	7	4
7	9	7	4

Table 2
Distance function and technologies.

Firms	\mathbb{B} -VRS	\mathbb{B} -CRS	C-VRS	C-CRS	FDH
1	1	4/3	1	4/3	1
2	1	1	1	1	1
3	4/3	4/3	4/3	4/3	1
4	1	1	1	1	1
5	7/6	7/6	7/6	7/6	7/6
6	8/7	8/7	1	8/7	1
7	9/8	9/8	9/8	9/7	9/8

Example 4.5.3. The following data sample, with $m = 1$, $n = 2$ and $l = 7$, can be found in Färe, Grosskopf and Lovell [7] (see Table 1).

For $k, k' \in \{1, \dots, l\}$ let $\alpha_k(x^{k'}) = a_{k',k}$ and let A be the $l \times l$ matrix whose coefficients are the numbers $a_{k',k}$.

With the data from Table 1 we obtain:

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 & 7/2 & 4 & 9/2 \\ 1 & 1 & 2 & 3 & 7/2 & 4 & 9/2 \\ 1/2 & 1/3 & 1 & 3/2 & 7/4 & 2 & 9/4 \\ 1/3 & 1/3 & 2/3 & 1 & 7/6 & 4/3 & 3/2 \\ 2/7 & 2/7 & 4/7 & 6/7 & 1 & 8/7 & 9/7 \\ 1/4 & 1/4 & 4/8 & 3/4 & 7/8 & 1 & 9/8 \\ 2/9 & 2/9 & 4/9 & 2/3 & 7/9 & 8/9 & 1 \end{pmatrix}.$$

The values of the Shephard distance function for the *DEA* and *FDH* estimations are listed in Table 2 where \mathbb{B} stands for \mathbb{B} -convex and C for convex (see Table 2).

The results obtained under a \mathbb{B} -convexity assumption are no less than those obtained in the convex and *FDH* cases. One can easily check that the inefficiency score is greater in the *CRS* models than in the *VRS* models. This is especially true in \mathbb{B} -convex versions. Notice that firms 2 and 4 are efficient for all types of estimations. Moreover, the *FDH* estimation being the minimal extrapolation of the “observed” data-set yields the largest number of efficient firms. Finally, notice that firm 6 lies on the frontier of the convex *VRS* model though it is inefficient in the \mathbb{B} -convex model.

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